State Feedback Robust Tracking Controller 
Based on Preview-$H_{\infty}$ Control Approach

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Abstract
This paper proposes a novel design method of a state feedback robust tracking controller based on 
preview-$H_{\infty}$ (infinity) control approach. The name of preview-$H_{\infty}$ control in this paper implies the 
combination between preview control theory and $H_{\infty}$ control theory. By assuming that the radar target future 
travel path can be predicted for certain number of sampling period, the future information of reference signal 
can be utilized to construct a tracking controller with better tracking performance. On the other hand, the 
$H_{\infty}$ control theory has been used to make the control system more robust against internal and external 
disturbances including wind force. The formulation of state feedback preview-$H_{\infty}$ control law has been 
carried out by the use of the maximum principle and differential game theory. Moreover, the relation between 
the cost function and $H_{\infty}$-norm criterion has been analyzed.

Key words: state feedback, robust, tracking, controller, preview, $H_{\infty}$ control, radar, target, travel, path.

1. Introduction

A tracking radar should follows the position of one or more objects in space, ignoring the content of the space not occupied by
the target(s) being tracked. In Single Target Track (STT) the radar follows a single object and ignores all others. In Multi Target Track (MTT) the radar continuously monitors the position of several targets, with each target sampled many times per second. Effectively, one radar performs the function of as many tracking radars as there are targets being tracked. Multi-target tracking requires that the antenna’s beam position be changeable essentially instantaneously, and this is normally possible only with electronically scanned antennas. For true tracking to take place, the target(s) must be sampled at the Nyquist rate for the track servo bandwidth and target maneuvering bandwidths [1].

By the use of the so called $\alpha - \beta$ filter whose parameters are tuned using the Kalman filter algorithm the radar target travel path can be predicted [2]. When the radar target travel path undergoes totally unpredictable high-G maneuvers it is often desired to use multiple tracking gates [3].

By assuming that the radar target future travel path can be estimated for certain number of sampling period during operation, the future
information of the reference signal can be utilized to construct a tracking control system with better tracking performance. The control method which incorporates the future information is well known as preview control. So far the preview control has been analyzed in the framework of Linear Quadratic Regulator (LQR) or Linear Quadratic Gaussian (LQG) control theory, and the preview control system has been designed to optimize a cost function of Linear Quadratic form which contains certain state variables, reference signals, and control input needed. It has been known that, compared with feedback LQR/LQG, the preview control makes further reduction of the minimum value achievable by the cost function. Also the preview control results in a zero-phase control law has been derived in the continuous time domain [4]. In spite of these advantages, the preview control has the lack in that it does not take into consideration the effects of uncertain disturbance. The disturbances may be present in two types, i.e., internal disturbance due to unmodeled dynamics or parametric variation, and external disturbance such as friction forces, wind forces, and so on. Using $H_\infty$ control theory, these disturbances can be taken into consideration in the design stage of the control system. The specifications such as stability robustness against internal disturbance and low sensitivity against external disturbance are reduced to a criterion in terms of $H_\infty$ norm by means of appropriate weighting functions [7], [8]. However, the feedback $H_\infty$ control uses only information corresponding to the present state value, and the limitations such phase-delay will probably be encountered.

This paper presents a novel method for designing a state feedback robust tracking controller based on preview- $H_\infty$ control approach which is a combination of preview control and $H_\infty$ control. The analytical formulation of the state feedback preview-$H_\infty$ control law has been derived in the continuous time domain.

2. Optimal Preview Control

In this section, for the sake of simplicity, a linear dynamic system as shown in state space equation (1) is considered. Here, $x_1, x_2$ are the state variables of the plant to be controlled, and $x_\epsilon$ is a manipulated state variable which is obtained by integrating the error signal between $x_2$ and a reference signal $r$. In this paper, the integral action is incorporated to ensure zero steady-state error. $u$ is the control input. Evaluation equation is expressed in equation (4) where $(z_1, z_{21}, z_{22})$ are evaluation signals and $(\eta_1, \eta_2)$ are weighting coefficients which should be adjusted during the design of the controller. During the formulation of the optimal preview control law disturbance $w_0$ due to uncertainty of either the plant or the environment which is not a priori known is not used.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_\epsilon \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\epsilon \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \quad (1)$$

Or simply,

$$\dot{x} = Ax(t) + Br(t), \quad (3)$$

where: $x(t_0) = x_0$.

$$\begin{bmatrix} z_1 \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\epsilon \end{bmatrix} + \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix} u \quad (4)$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_\epsilon \end{bmatrix} + \begin{bmatrix} \xi \\ 0 \end{bmatrix} u \quad (5)$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \xi \\ 0 \end{bmatrix} u \quad (6)$$

$$z(t) = Cx(t) + Du(t). \quad (7)$$

Here, the following cost function to be minimized is considered,

$$J = \frac{1}{2} z_2'(t_f)Sz_2(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[ z_2'(t)Qz_2(t) + u'(t)Ru(t) \right] dt, \quad (8)$$

where: $S, Q, R = \xi'\xi$ are positive definite matrices. Let the Hamiltonian be defined as follows:
\[ H(x,u,\lambda,t) = \frac{1}{2} z'_i(t)Q z_i(t) + \frac{1}{2} u'(t)Ru(t) + \lambda'(t) [A x(t) + B_u u(t) + B_r r(t)] \]

where \( \lambda(t) \) is the Lagrange multiplier. Using the maximum principle, in order to minimize the cost function, the following must hold \([5]\),

\[ \frac{\partial H}{\partial u} = 0 \]  

Which yields,

\[ u = -R^{-1}B'_i\lambda(t), \]

And

\[ \frac{\partial H}{\partial x} = \lambda = C'Q_c x(t) + A' \lambda(t) \]

With the terminal condition,

\[ \lambda(t_f) = \frac{\partial}{\partial x(t_f)} \left( \frac{1}{2} z'_i(t_f)S z_i(t_f) \right) = C'_f S C_2 x(t_f). \]  

To determine the control law, it is assumed that,

\[ \lambda(t) = P(t)x(t) + q(t). \]  

From equations (3), (11), (12), and (14), the following differential equations are obtained,

\[ \dot{P}(t) + P(t)A + A'P(t) = -P(t)B_iR^{-1}B_i'P(t)0 + C'_Q Q_c = 0 \]  

\[ \dot{q}(t) + [A - B_i R^{-1} B'_i P(t)]' q(t) + P(t)B_r r(t) = 0 \]

where the boundary conditions are,

\[ P(t_f) = C'_f S C_2 = P_{tf} \]  

\[ q(t_f) = 0 \]

The control law is then given by,

\[ u(t) = -R^{-1}B'_i [P(t)x(t) + q(t)] \]

\[ = -R^{-1}B'_i P(t) x(t) - R^{-1}B'_i q(t) \]

\[ = u_b + u_f \]

where \( P(t) \) and \( q(t) \) are the solution of equation (15) and equation (16), respectively. The total control input is composed by feedback control input \( u_b \) and feed-forward control input \( u_f \).

Therefore, the closed loop system can be illustrated as in figure 1. \( P \) is the dynamical system to be controlled, \( K = -r^{-1}B'_i P(t) \), and \( F_f \) represents the reference signal processor calculating \( q(t) \) according to equation (16).

3. Preview - \( H_{\infty} \) Control

3.1. Reference Signal a Priori Known for Total Working Time \([t_0, t_f]\)

Taking into consideration the uncertain disturbance \( w_i \) and setting \( \xi = 1 \), the state-space equation (3) and the evaluating equation (7) can be expressed by the following equations:

\[ \dot{x} = Ax(t) + B_1 u(t) + B_r r(t) + B_w w_i(t) \]  

where \( x(t_0) = 0 \), and

\[ z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \]

Equation (20) can be expressed as follows

\[ \dot{x} = Ax(t) + B_1 u(t) + B_r r(t) \]  

\[ \dot{x} = Ax(t) + B r(t) + B_w w_i(t) \]  

Suppose that the reference signal \( r(t) \) is known during the total working time \([t_0, t_f]\), and let a cost function be defined by,

\[ J_f = \frac{1}{2} z_i(t_f) S z_i(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[ z'_i z_i + \gamma^2 \left[ \begin{array}{c} w_i \\ 0 \end{array} \right] \right] dt \]

\[ = \frac{1}{2} z'_i(t_f) S z_i(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[ z'_i z_i + v^2 R v \right] dt \]

where \( \gamma \) is a given positive value.

Analog to the previous section, the cost function \( J_{[b_i, t_f]} \) is optimized by

\[ w_i(t) = \gamma^2 B_w' P(t) x(t) + \gamma^2 B_r' q(t) \]

\[ u(t) = -B'_i P(t) x(t) - B'_i q(t). \]
Let define a Lyapunov function of the energy as:

\[ V = x^T P x, \]

Then the following equation must hold,

\[ \int_0^T [x^T \bar{P} x + x^T \bar{P} x + x^T \bar{P} x] dt - x^T \bar{P} x |_0^T = 0 \]

Substituting equation (23) into equation (37) the following equation is obtained,

\[ \int_0^T [x' (\bar{P} + A \bar{P} + \bar{A}) x + (x \bar{P} B, u + u' B_1 \bar{P} x) + (x \bar{P} B, w_1 + \bar{w}' B_1 \bar{P} x)] dt \]

\[ - x^T \bar{P} x |_0^T + \int_0^T (x^T \bar{P} B, r + r' B_1 \bar{P} x) dt = 0. \] (38)

Since adding equation (38) into the cost function of equation (34) does not change the value, the cost function can be expanded as follows,

\[ J_{[r,F]} = x_f^T F x_f - x^T \bar{P} x |_0^T + \int_0^T [x' \bar{P} B, u + u' B_1 \bar{P} x] dt - \gamma^2 \int_0^T [w_1 - \gamma^2 B_1 \bar{P} x]' [w_1 - \gamma^2 B_1 \bar{P} x] dt \]

\[ + \int_0^T (x^T \bar{P} B, r + r' B_1 \bar{P} x) dt. \] (39)

where \( \alpha = \bar{P} + A \bar{P} + \bar{P} A + C_2' S C_2 \).

By substituting \( \bar{P} \) of equation (28) into equation (39) and recalling that \( x(0) = 0 \) yields

\[ J_{[r,F]} = \int_0^T [u + B_1 \bar{P} x]' [u + B_1 \bar{P} x] dt - \gamma^2 \int_0^T [w_1 - \gamma^2 B_1 \bar{P} x]' [w_1 - \gamma^2 B_1 \bar{P} x] dt \]

\[ + \int_0^T (x^T \bar{P} B, r + r' B_1 \bar{P} x) dt \] (40)

If \( r(t) = 0 \), the third term in the right side of equation (40) become zero. In this case the saddle point is given by

\[ u(t) = u^*(t) = -B_1 \bar{P}(t) x(t) \]

\[ w(t) = w^*(t) = \gamma^2 B_1 \bar{P}(t) x(t) \]

where \( u^*, w^* \) are optimum solution for controlled input and disturbance, respectively. Here, \( w^* \) can be interpreted as the worst disturbance.

Therefore, the cost function above can be expressed by

\[ J_{[r,F]} = \| u - u^* \|_2 |_{2[0,T]} - \gamma^2 \| w - w^* \|_2 |_{2[0,T]} \] (43)
for any controlled input $u$ and any disturbance $w$. For the optimum control input, equation (43) means that

$$\|T_{zw}\|_{[0,T]} < \gamma.$$  
(44)

If $T$ is assumed to be infinity, then $\hat{P} = 0$, and the $H_\infty$ norm criterion of equation (33) or equation (34) is satisfied.

When $r(t) \neq 0$, by substituting the optimum $w_f(t)$ in equation (26) and $u$ in equation (27) into the cost function in equation (40), the following is obtained

$$J_{[T,F]opt} = \int_0^T [(q^{-1}(t)(B_1 B'_1 - B_w B'_w) \bar{q}(t))dt$$

$$+ \int_0^T [x^\top B_r r + B'_r \bar{P} x] dt$$  
(45)

According to the induced norm, the following relation are obtained

$$\|T_{zw}\|_{[0,T]} = J_{[T,F]opt} - x'_T F_{XT}$$  
(46)

$$\|T_{zw}\|_{[0,T]} = J_{[\infty,F]opt} - x'_\infty F_{X\infty}$$  
(47)

Therefore when the reference signal $r(t)$ is present, the $H_\infty$-norm criterion of the transfer function from the disturbance to the evaluation signal is affected by it.

### 3.2 Reference Signal Previewed for a Certain Short Interval of Time $[t_0,t_0 + \tau]$

In this case, it is assumed that the reference signal $r(t)$ is known only during a certain interval of time $\tau$ which is much lesser than the total working time $T$ which is assumed to be long enough. During the interval $[t_0,t_0 + \tau]$ the following cost function is considered

$$J_1 = \int_{t_0}^{t_0 + \tau} [z(t)'z(t) - \gamma^2 w_l(t)w(t)]dt.$$  
(48)

Since for the rest interval of time $[t_0,t_0 + \tau]$ the reference signal is unknown, it is considered as uncertain disturbance along with $w_l(t)$.

Assuming that $T$ is infinity, during this interval the following cost function is considered

$$J_2 = \int_{t_0 + \tau}^\infty [z(t)'z(t) - \gamma^2 w(t)'w(t)]dt.$$  
(49)

Where

$$w = \begin{bmatrix} r(t) \\ w_l(t) \end{bmatrix}.$$  
(50)

Therefore, total cost function to be optimized is

$$J_{tot} = J_1 + J_2.$$  
(51)

From linear Optimal control Theory it is well known that the optimum value of the cost function $J_2$ is

$$J_2 = x'(t_0 + \tau)\hat{P}x(t_0 + \tau).$$  
(52)

where $\hat{P}$ is the solution of the following Algebraic Riccati Equation

$$\hat{P}A + A^\top\hat{P} + C_r C_r^\top\hat{P}B_r$$

$$- \hat{P}[B_1 B'_1 - \gamma^2 (B_r B'_r + B_w B'_w)]\hat{P} = 0$$  
(53)

From equations (49), (51), (52) the cost function $J_{tot}$ can be expressed as:

$$J_{tot} = x'(t_0 + \tau)\hat{P}x(t_0 + \tau) +$$

$$\int_{t_0}^{t_0 + \tau} [z(t)'z(t) - \gamma^2 w_l(t)w_l(t)]dt.$$  
(54)

Since the cost function of equations (54) and (35) are similar with the only difference in their terminal conditions, the control law which optimizes the cost function $J_{tot}$ of equations (54) subject to the constraint of equation (20), is also given by equation (27) with $\hat{P}(t)$ calculated from equation (28) and $\bar{q}(t)$ calculated from equation (29).

More over, since the cost function of equation (54) and (35) are similar with the only difference in their terminal conditions, the analysis of $H_\infty$-norm of the transfer function from the disturbance $w_l(t)$ to the evaluation signal $z(t)$ can be carried out in similar way as in the previous subsection.

### 4. Conclusions and Discussions

The preview- $H_\infty$ control law has been derived which has the feedback and feed forward parts. The control law is given by solving a Riccati equation and by calculating a differential equation of the signal used in the feed forward part. Concerning the prediction of future value of reference signal, two results have been obtained, i.e., reference signal previewed for total working time $[t_0,t_f]$, and reference signal previewed for a certain short interval of time $[t_0,t_0 + \tau]$.

From the analysis of $H_\infty$-norm criterion it has been found that the reference signal has an effect on the value of the $H_\infty$-norm of the transfer function from the disturbance to the evaluation signal.
References